Nonparametric State-Price Density Estimation using High Frequency Data

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Introduction

- In a complete market with no arbitrages, there exists a unique measure $\mathbb{Q} \sim \mathbb{P}$ such that
$$\xi_t = E^{\mathbb{Q}}\left(e^{-r(T-t)}\xi_T \middle| \mathcal{F}_t\right)$$
for any traded payoff ξ_T. The density of $\mathbb{Q}$ is the state-price density (SPD).
    - a European call option has no-arbitrage price
    $$C_t = e^{-r(T-t)} \int_K^{\infty} (S_T - K) f_t^{\mathbb{Q}}(S_T) dS_T$$
    - Breeden and Litzenberger (1978):
    $$\frac{\partial^2 C_t}{\partial K^2} = \mathrm{e}^{-r(T-t)} f_{t,S_T}^{\mathbb{Q}}(K)$$
    - Applications:
        1. option pricing
        2. risk management
        3. empirical pricing kernels/risk aversion

- The call pricing function can be modelled by the nonparametric regression

$$C_t = m(\tau, F_t, K_t) + \epsilon_t.$$

  Methods can be categorized by

  - smoothing method: kernel smoothing, smoothing splines
  - implementing shape constraints
  - further dimension reductions. Two issues:
    1. nonstationarity of the price level F_t
    2. discreteness of strike price K_t

    Potential solution: assume homogeneity (Chen and Xu, 2014)

$$m(\tau, F, K) = Fm(\tau, 1, K/F) = F\tilde{m}(\tau, M),$$

    and treat 'moneyness' $M = K/F$ as stationary and continuous

    ★ holds if the distribution of the return of F is independent from its level (Merton, 1973)

Methodology

- Call pricing function commonly depends on t only via time-to-maturity $\tau = T - t$.
  - implies that, for each maturity, SPD is constant over time
  - yet call prices $(C_t)_t$ are expectations wrt filtration $(\mathcal{F}_t)_t$
- With high frequency data, we may allow the SPD to vary within sample. Consider the time-varying regression

$$C_t = m(t, T, F_t, K_t) + \epsilon_t,$$

  - dimension reductions: assume homogeneity, and one maturity date T

$$\tilde{C}_t = \tilde{m}(t, M_t) + \tilde{\epsilon}_t,$$

where $\tilde{\cdot} = \cdot / F_t$, and $E(\tilde{\epsilon}_t | M_t) = 0$

- Nadaraya-Watson kernel smoother:

$$\hat{\tilde{m}}(t,x) = \frac{\sum_{i=1}^{n} K_{h_t}(t - t_i) K_{h_M}(x - M_{t_i}) \tilde{C}_{t_i}}{\sum_{i=1}^{n} K_{h_t}(t - t_i) K_{h_M}(x - M_{t_i})},$$

  for a kernel $K(\cdot)$, and bandwidths h_t and h_M

- Asymptotic properties of $\hat{\tilde{m}}(t,x)$ for $T \to \infty$ studied by Vogt (2012)
  - adapt to infill asymptotics with $t_0 < ... < t_n < T$ and $n \to \infty$
- How to measure time distance in $K_{h_t}(t_j - t_i)$?
  - investigate time-deformation approach $K_{h_t}(g_n(t_j) - g_n(t_i))$, with e.g.
    - ★ transaction count measure $g_n(t_i) = i/n$
    - ★ volume-weighted measure $g_n(t_i) = \frac{CumVol_{t_i}}{CumVol_{t_n}}$
  - allow one-sided kernels for time-dimension, i.e. $K_{h_t} : \mathbb{R}^+ \to \mathbb{R}^+$

- Test the Merton (1973) condition using

$$f^{\mathbb{Q}}_{t,F_T/F_t}(x) = \frac{\partial^2 \tilde{m}(t,M)}{\partial M^2}\bigg|_{M=x}$$

    - regress moments of $\hat{f}^{\mathbb{Q}}_{t,F_T/F_t}$ on intraday levels of F_t
    - need independence only to hold locally in time
- Microstructure (and synchronization) noise in futures price $F_t = F^{\circ}_t + \eta_t$ creates error-in-variables problem
    - test: run separate regressions with bid and ask prices $a_t \leq F^{\circ}_t \leq b_t$, and check if difference is significant

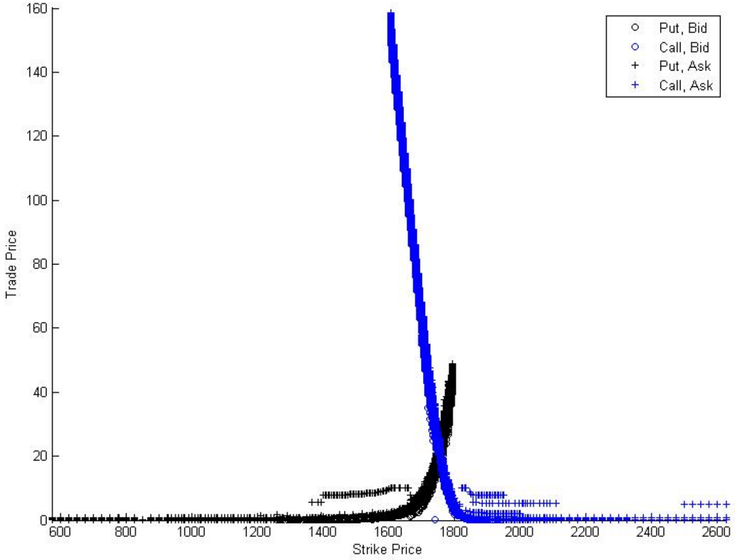

Figure: Best bid and ask prices of call and put options plotted against strike price, for November 1, 2013.

Summary

- Use high frequency data to estimate short-run dynamics of SPDs
  - apply time-varying regression model
  - test homogeneity
  - investigate measurement error