Nonparametric State-Price Density Estimation using High Frequency Data

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Introduction

- In a complete market with no arbitrages, there exists a unique measure \( \mathbb{Q} \sim \mathbb{P} \) such that
  \[
  \xi_t = \mathbb{E}^{\mathbb{Q}} \left( e^{-r(T-t)} \xi_T | \mathcal{F}_t \right)
  \]
  for any traded payoff \( \xi_T \). The density of \( \mathbb{Q} \) is the state-price density (SPD).
  - a European call option has no-arbitrage price
    \[
    C_t = e^{-r(T-t)} \int_{-\infty}^{\infty} (S_T - K) f_t^Q(S_T) dS_T
    \]
  - Breeden and Litzenberger (1978):
    \[
    \frac{\partial^2 C_t}{\partial K^2} = e^{-r(T-t)} f^Q_{t,S_T}(K)
    \]
  - Applications:
    1. option pricing
    2. risk management
    3. empirical pricing kernels/risk aversion
The call pricing function can be modelled by the nonparametric regression

\[ C_t = m(\tau, F_t, K_t) + \epsilon_t. \]

Methods can be categorized by

- smoothing method: kernel smoothing, smoothing splines
- implementing shape constraints
- further dimension reductions. Two issues:
  1. nonstationarity of the price level \( F_t \)
  2. discreteness of strike price \( K_t \)

Potential solution: assume homogeneity (Chen and Xu, 2014)

\[ m(\tau, F, K) = Fm(\tau, 1, K/F) = F\tilde{m}(\tau, M), \]

and treat ‘moneyness’ \( M = K/F \) as stationary and continuous

\* holds if the distribution of the return of \( F \) is independent from its level (Merton, 1973)
Call pricing function commonly depends on $t$ only via time-to-maturity $\tau = T - t$.

- implies that, for each maturity, SPD is constant over time
- yet call prices $(C_t)_t$ are expectations wrt filtration $(\mathcal{F}_t)_t$

With high frequency data, we may allow the SPD to vary within sample. Consider the time-varying regression

$$C_t = m(t, T, F_t, K_t) + \epsilon_t,$$

- dimension reductions: assume homogeneity, and one maturity date $T$

$$\tilde{C}_t = \tilde{m}(t, M_t) + \tilde{\epsilon}_t,$$

where $\tilde{\cdot} = \cdot / F_t$, and $E(\tilde{\epsilon}_t | M_t) = 0$
Nadaraya-Watson kernel smoother:

\[ \hat{m}(t, x) = \frac{\sum_{i=1}^{n} K_{ht}(t - t_i) K_{hM}(x - M_{t_i}) \tilde{C}_{t_i}}{\sum_{i=1}^{n} K_{ht}(t - t_i) K_{hM}(x - M_{t_i})}, \]

for a kernel \( K(\cdot) \), and bandwidths \( h_t \) and \( h_M \)

Asymptotic properties of \( \hat{m}(t, x) \) for \( T \to \infty \) studied by Vogt (2012)

- adapt to infill asymptotics with \( t_0 < \ldots < t_n < T \) and \( n \to \infty \)

How to measure time distance in \( K_{ht}(t_j - t_i) \)?

- investigate time-deformation approach \( K_{ht}(g_n(t_j) - g_n(t_i)) \), with e.g.
  - transaction count measure \( g_n(t_i) = i/n \)
  - volume-weighted measure \( g_n(t_i) = \frac{CumVol_{t_i}}{CumVol_{t_n}} \)

- allow one-sided kernels for time-dimension, i.e. \( K_{ht} : \mathbb{R}^+ \to \mathbb{R}^+ \)
Test the Merton (1973) condition using

\[ f_{t,F_t/F_t}^Q(x) = \left. \frac{\partial^2 \tilde{m}(t, M)}{\partial M^2} \right|_{M=x} \]

- regress moments of \( \hat{f}_{t,F_t/F_t}^Q \) on intraday levels of \( F_t \)
- need independence only to hold locally in time

Microstructure (and synchronization) noise in futures price \( F_t = F_t^\circ + \eta_t \) creates error-in-variables problem
- test: run separate regressions with bid and ask prices \( a_t \leq F_t^\circ \leq b_t \), and check if difference is significant
Figure: Best bid and ask prices of call and put options plotted against strike price, for November 1, 2013.
Summary

- Use high frequency data to estimate short-run dynamics of SPDs
  - apply time-varying regression model
  - test homogeneity
  - investigate measurement error